

# Vanishing of cohomology and property (T) for groups acting on weighted simplicial complexes

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## Abstract

We extend Ballmann and Świątkowski's work on cohomology of groups acting on simplicial complexes and provide further vanishing results of cohomologies and of property (T). In particular, we give a new criterion for property (T) for groups acting on an  $n$ -dimensional simplicial complex.

## 1 Introduction

For a finite graph  $L$  with a set of vertices  $V_L$ , the Laplacian of the graph  $\Delta^+$  is an operator on the space of real valued functions on  $V_L$  which is defined as

$$\Delta^+ f(v) = f(v) - \frac{1}{m(v)} \sum_{u \sim v} f(u)$$

where  $m(v)$  is the valance of  $v$  and  $u \sim v$  means that there is an edge connecting  $u$  and  $v$ . The Laplacian is a positive operator and we denote by  $\lambda(L)$  its smallest positive eigenvalue. One can generalize the definition of the Laplacian in two ways - first one can put weights on the edges and second one can generalize the definition the Laplacian will be defined for a simplicial complex  $X$  of any dimension. For such a complex the (weighted) Laplacian is again a positive operator and we denote by  $\lambda(X)$  its smallest positive eigenvalue.

Ballmann and Świątkowski in [BŚ97] and independently Żuk's in [Żuk96] gave criteria for the vanishing of cohomologies of a group  $\Gamma$  acting on a simplicial complex  $X$ , by considering the values of  $\lambda$  for the links of  $X$  (in both cases, the authors were relying on the previous work of Garland [Gar73]).

In [DJ00] the above results were generalized by Dymara and Januszkiewicz to a more general setting in which  $\Gamma$  isn't necessarily discrete but just locally compact and unimodular.

The case which maybe hold the most interest in the vanishing of the first cohomology, because this is equivalent to property (T).

In this paper, we shall generalize the criterion given in [BŚ97] to a criterion for weighted Laplacian and in this slightly more general setting we will prove new vanishing of cohomology results. Our results are not new only in the weighted setting, but are also in the setting given in [BŚ97]. Namely (in the setting of [BŚ97]), we will prove the following theorem:

**Theorem 1.1.** *Let  $X$  be a contractible simplicial complex of dimension  $n$ , such that all the links of  $X$  are connected, and such that there is an integer  $0 \leq r \leq n - 2$  such that the link of every  $r$ -simplex is finite. Let  $\Gamma$  be a locally compact and unimodular group acting cocompactly and properly discontinuously by automorphisms on  $X$ . For a simplex  $\tau$  of dimension  $k - 1 \geq r$ , denote by  $X_\tau$  the link of  $\tau$  which is a simplicial complex and denote by  $\lambda(X_\tau)$  the smallest positive eigenvalue of the (un-weighted) Laplacian on  $X_\tau$ .*

*Let  $r < k \leq n - 1$ , if  $\lambda(X_\tau) > \frac{k(n-k)}{k+1}$  for every simplex  $\tau$  of dimension  $k - 1 \geq r$ , then:*

1. *For every  $r < j \leq k$  we have  $H^j(\Gamma, \rho) = H^j(X, \rho) = 0$  for any unitary representation  $\rho$  of  $\Gamma$ .*
2.  *$\Gamma$  has property (T).*

The power of this theorem is that it allows us to prove property (T) (and the vanishing of the  $r + 1$  to  $n - 1$  cohomologies) just by calculating the Laplacian's smallest positive eigenvalue for links of simplices of dimension  $n - 2$  (in those cases the links are graphs, so the eigenvalues are easier to compute).

**Remark 1.2.** *Theorems similar to the one stated above were proven in several articles (other than [BŚ97] and [Żuk96] that were already mentioned). We will not give a detail account about all those articles, but just mention a few important ones:*

1. *[Gar73] considered the case of a group  $\Gamma$  acting on a building whose all of its vertices has finite links. In this case, [Gar73] showed that for every  $0 \leq k \leq n - 1$  there is a  $\varepsilon > 0$  such that if all the positive Laplacian eigenvalues of links of dimension 1 are strictly larger than  $1 - \varepsilon$  then  $H^k(\Gamma, \rho) = 0$ . But the nature of this result was less quantitative and more asymptotic - [Gar73] showed that if the smallest positive Laplacian eigenvalue of the links of dimension 1 tend to 1 then all the cohomologies vanish.*
2. *[DJ02] considers the case of a group acting on a simplicial complex where the fundamental domain is a single simplex and there is an integer  $0 \leq r \leq n - 2$  such that the link of every  $r$ -simplex is finite the links are finite. In this case, [DJ02] proves a vanishing result for the first to the  $n - r - 1$  cohomologies relaying on the smallest positive Laplacian eigenvalue of links of dimension 1. The proof in [DJ02] uses totally different methods than of Garland's (and of this paper). However, the eigenvalue*

the Laplacian needed for such a vanishing result in [DJ02] is quite large: the smallest positive eigenvalue should be strictly larger than  $1 - \frac{13}{28^n}$  to ensure the vanishing of the cohomologies (compared to the theorem stated above, where it should be strictly larger than  $\frac{n-1}{n}$ ).

3. [Kas11] considers a similar case to case considered in [DJ02]. In this case, [Kas11] proves that if the smallest positive eigenvalue is larger than  $\frac{n-1}{n}$ , then the group has property (T). In fact, [Kas11] proves a little more general criterion which takes into account the interplay between different 1 dimensional links. However, the proof of [Kas11] relies on the fact that the fundamental domain is a single simplex and does not generalize to our case where the fundamental domain needs only to be compact.

**Structure of the paper.** Section 2 is devoted to introducing the framework developed in [BS97] with some generalizations (weights) and additions (restriction), section 3 contains three subsections - the first proves a result similar to the one given in [BS97] in the weighted setting, the second proves a stronger vanishing result in as in the theorem stated above and the third proves property (T) (as in the theorem stated above).

## 2 Framework

Here we introduce a slight generalization of the framework constructed in [BS97] and [DJ00]. We should note that this generalization was already considered by Wang in [Wan98] but for completeness we will prove all the propositions that differ (by a constant) from [BS97] and [DJ00].

Throughout this paper  $X$  is a simplicial complex of dimension  $n$  such that all the links of  $X$  are connected and there is a constant integer  $0 \leq r \leq n-2$  such that all the links of simplices of dimension  $r$  are finite. Also  $\Gamma$  is a locally compact, properly discontinuous, unimodular group of automorphisms of  $X$  acting cocompactly on  $X$  and  $\rho$  is a unitary representation of  $\Gamma$  on a complex Hilbert space  $H$ .

### 2.1 general settings

Following [BS97] we introduce the following notations:

1. For  $0 \leq k \leq n$ , denote by  $\Sigma(k)$  the set of ordered  $k$ -simplices (i.e.  $\sigma \in \Sigma(k)$  is an ordered  $k+1$ -tuple of vertices) and choose a set  $\Sigma(k, \Gamma) \subseteq \Sigma(k)$  of representatives of  $\Gamma$ -orbits.
2. For a simplex  $\sigma \in \Sigma(k)$ , denote by  $\Gamma_\sigma$  the stabilizer of  $\sigma$  and by  $|\Gamma_\sigma|$  the measure of  $\Gamma_\sigma$  with respect to the Haar measure.

3. For  $0 \leq k \leq n$ , denote by  $C^k(X, \rho)$  the space of simplicial  $k$ -cochains of  $X$  which are twisted by  $\rho$ , that is,  $\phi \in C^k(X, \rho)$  is an alternating map on ordered  $k$ -simplices of  $X$  with values in  $H$  such that

$$\forall \gamma \in \Gamma, \forall x \in \Sigma(k), \rho(\gamma)\phi(x) = \phi(\gamma x)$$

The following proposition is taken from [BS97], [DJ00]:

**Proposition 2.1.** [BS97, Lemma 1.3], [DJ00, Lemma 3.3] For  $0 \leq l < k \leq n$ , let  $f = f(\tau, \sigma)$  be a  $\Gamma$ -invariant function on the set of pairs  $(\tau, \sigma)$ , where  $\tau$  is an ordered  $l$ -simplex and  $\sigma$  is an ordered  $k$ -simplex with  $\tau \subset \sigma$ . Then

$$\sum_{\sigma \in \Sigma(k, \Gamma)} \sum_{\substack{\tau \in \Sigma(l) \\ \tau \subset \sigma}} \frac{f(\tau, \sigma)}{|\Gamma_\sigma|} = \sum_{\tau \in \Sigma(l, \Gamma)} \sum_{\substack{\sigma \in \Sigma(k) \\ \tau \subset \sigma}} \frac{f(\tau, \sigma)}{|\Gamma_\tau|}$$

The reader should note, that from now on we will use the above proposition to change the order of summation without mentioning it explicitly.

**Definition 2.2.** A weight on  $X$  is an equivariant function  $m : \bigcup_{k \geq r} \Sigma(k) \rightarrow \mathbb{R}^+$  such that:

1. For every  $\tau = (v_0, \dots, v_k)$  and for every permutation  $\sigma \in S_k$  we have  $m((v_0, \dots, v_k)) = m((v_{\sigma(0)}, \dots, v_{\sigma(k)}))$ .
2. For every  $r \leq k \leq n-1$  there is a  $C_k$  such that for every  $\tau \in \Sigma(k)$  we have the following equality

$$\sum_{\sigma \in \Sigma(k+1), \tau \subset \sigma} m(\sigma) = (k+2)! C_k m(\tau)$$

Where  $\tau \subset \sigma$  means that all the vertices of  $\tau$  are contained in  $\sigma$  (with no regard to the ordering).

**Example 2.3.** In [BS97] the function  $m$  was defined as: for every  $\tau \in \Sigma(k)$ ,  $m(\tau)$  is the number of (unordered) simplices of dimension  $n$  that contain  $\tau$ . In that case,  $m$  is the constant function 1 on  $\Sigma(n)$  and  $C_k = n - k$ .

**Remark 2.4.** There is a lot of freedom in our definition of the weight function. Without loss of generality, one can always normalize the weight function such that  $C_k = 1$  for all  $k$  (and this is the setting used in [Wan98]). It is obvious that in the normalized case the function  $m$  is determined by its values on  $\Sigma(n)$ . We chose not to normalize the weight function in this paper as a matter of convenience and so that the reader could easily compare our results to those proven in [BS97].

For  $k \geq r$  define an Hermitian form on  $C^k(X, \rho)$  as

$$\langle \phi, \psi \rangle := \sum_{\sigma \in \Sigma(k, \Gamma)} \frac{m(\sigma)}{(k+1)! |\Gamma_\sigma|} \langle \phi(\sigma), \psi(\sigma) \rangle$$

To distinguish the norm of  $C^k(X, \rho)$  from the norm of  $H$  we will use  $|\cdot|$  for the norm of  $H$  (i.e.  $\langle \phi(\sigma), \phi(\sigma) \rangle = |\phi(\sigma)|^2$ ).

For  $r \leq k < n$ , the differential  $d : C^k(X, \rho) \rightarrow C^{k+1}(X, \rho)$  is given by

$$d\phi(\sigma) := \sum_{i=0}^{k+1} (-1)^i \phi(\sigma_i), \quad \sigma \in \Sigma(k+1)$$

Where  $\sigma_i = (v_0, \dots, \hat{v}_i, \dots, v_{k+1})$  for  $(v_0, \dots, v_{k+1}) = \sigma \in \Sigma(k+1)$ .

Denote

$$\Delta^+ = \delta d : C^k(X, \rho) \rightarrow C^k(X, \rho)$$

$$\Delta^- = d\delta : C^k(X, \rho) \rightarrow C^k(X, \rho)$$

and  $\Delta = \Delta^+ + \Delta^-$ .

Define the  $k$ -th cohomology as

$$H^k(X, \rho) = \ker(d : C^k(X, \rho) \rightarrow C^{k+1}(X, \rho)) / \text{im}(d : C^{k-1}(X, \rho) \rightarrow C^k(X, \rho))$$

**Proposition 2.5.** 1. (equivalent to [BS97, Proposition 1.5]) The differential  $d$  is a bounded operator.

2. (equivalent to [BS97, Proposition 1.6]) The adjoint operator of  $d$ , denoted by  $\delta : C^{k+1}(X, \rho) \rightarrow C^k(X, \rho)$  is

$$\delta\phi(\tau) = \sum_{\substack{v \in \Sigma(0) \\ v\tau \in \Sigma(k+1)}} \frac{m(v\tau)}{m(\tau)} \phi(v\tau), \quad \tau \in \Sigma(k)$$

where  $v\tau = (v, v_0, \dots, v_k)$  for  $\tau = (v_0, \dots, v_k)$

3. (equivalent to [BS97, Corollary 1.7]) For  $\phi \in C^k(X, \rho)$ ,  $k > r$  and  $\sigma \in \Sigma(k)$ ,

$$\Delta^+ \phi(\sigma) = \delta d\phi(\sigma) = C_k \phi(\sigma) - \sum_{\substack{v \in \Sigma(0) \\ v\tau \in \Sigma(k+1)}} \sum_{0 \leq i \leq k} (-1)^i \frac{m(v\sigma)}{m(\sigma)} \phi(v\sigma_i)$$

*Proof.* 1. For every  $\phi \in C^k(X, \rho)$  we have

$$\begin{aligned} \|d\phi\|^2 &= \sum_{\sigma \in \Sigma(k+1, \Gamma)} \frac{m(\sigma)}{(k+2)!|\Gamma_\sigma|} \left| \sum_{i=0}^{k+1} (-1)^i \phi(\sigma_i) \right|^2 \leq \\ &\leq \sum_{\sigma \in \Sigma(k+1, \Gamma)} \frac{m(\sigma)}{(k+2)!|\Gamma_\sigma|} (k+2) \sum_{i=0}^{k+1} |\phi(\sigma_i)|^2 \leq \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{\sigma \in \Sigma(k+1, \Gamma)} \frac{(k+2)m(\sigma)}{(k+2)!(k+1)!|\Gamma_\sigma|} \sum_{\tau \in \Sigma(k), \tau \subset \sigma} |\phi(\tau)|^2 \leq \\
&\leq \sum_{\tau \in \Sigma(k, \Gamma)} \frac{(k+2)|\phi(\tau)|^2}{(k+2)!(k+1)!|\Gamma_\tau|} \sum_{\sigma \in \Sigma(k+1), \tau \subset \sigma} m(\sigma) \leq \\
&\leq \frac{C_k}{(k+1)!} \sum_{\tau \in \Sigma(k, \Gamma)} \frac{m(\tau)|\phi(\tau)|^2}{(k+1)!|\Gamma_\tau|} = (k+2)C_k \|\phi\|^2
\end{aligned}$$

2. For  $\sigma \in \Sigma(k+1)$  and  $\tau \subset \sigma, \tau \in \Sigma(k)$  denote by  $[\sigma : \tau]$  the incidence coefficient of  $\tau$  with respect to  $\sigma$ , i.e. if  $\sigma_i$  has the same vertices as  $\tau$  then for every  $\psi \in C^k(X, \rho)$  we have  $[\sigma : \tau]\psi(\tau) = (-1)^i \psi(\sigma_i)$ . Take  $\phi \in C^{k+1}(X, \rho)$  and  $\psi \in C^k(X, \rho)$ , then we have

$$\begin{aligned}
\langle d\psi, \phi \rangle &= \sum_{\sigma \in \Sigma(k+1, \Gamma)} \frac{m(\sigma)}{(k+2)!|\Gamma_\sigma|} \langle \sum_{i=0}^{k+1} (-1)^i \psi(\sigma_i), \phi(\sigma) \rangle = \\
&= \sum_{\sigma \in \Sigma(k+1, \Gamma)} \frac{m(\sigma)}{(k+1)!(k+2)!|\Gamma_\sigma|} \langle \sum_{\substack{\tau \in \Sigma(k) \\ \tau \subset \sigma}} [\sigma : \tau] \psi(\tau), \phi(\sigma) \rangle = \\
&= \sum_{\sigma \in \Sigma(k+1, \Gamma)} \frac{m(\sigma)}{(k+1)!|\Gamma_\sigma|} \sum_{\substack{\tau \in \Sigma(k) \\ \tau \subset \sigma}} \langle \psi(\tau), \frac{[\sigma : \tau]m(\sigma)}{m(\tau)(k+2)!} \phi(\sigma) \rangle = \\
&= \sum_{\tau \in \Sigma(k, \Gamma)} \frac{m(\tau)}{(k+1)!|\Gamma_\tau|} \sum_{\substack{\sigma \in \Sigma(k+1) \\ \tau \subset \sigma}} \langle \psi(\tau), \frac{[\sigma : \tau]m(\sigma)}{m(\tau)(k+2)!} \phi(\sigma) \rangle = \\
&= \sum_{\tau \in \Sigma(k, \Gamma)} \frac{m(\tau)}{(k+1)!|\Gamma_\tau|} \langle \psi(\tau), \sum_{\substack{\sigma \in \Sigma(k+1) \\ \tau \subset \sigma}} \frac{[\sigma : \tau]m(\sigma)}{m(\tau)(k+2)!} \phi(\sigma) \rangle = \\
&= \sum_{\tau \in \Sigma(k, \Gamma)} \frac{m(\tau)}{(k+1)!|\Gamma_\tau|} \langle \psi(\tau), \sum_{\substack{v \in \Sigma(0) \\ v\tau \in \Sigma(k+1)}} \frac{m(v\tau)}{m(\tau)} \phi(v\tau) \rangle
\end{aligned}$$

3. For every  $\phi \in C^k(X, \rho)$  and every  $\sigma \in \Sigma(k)$  we have:

$$\delta d\phi(\sigma) = \sum_{\substack{v \in \Sigma(0) \\ v\sigma \in \Sigma(k+1)}} \frac{m(v\sigma)}{m(\sigma)} d\phi(v\sigma) =$$

$$\begin{aligned}
&= \sum_{\substack{v \in \Sigma(0) \\ v\sigma \in \Sigma(k+1)}} \frac{m(v\sigma)}{m(\sigma)} \phi(\sigma) - \sum_{\substack{v \in \Sigma(0) \\ v\sigma \in \Sigma(k+1)}} \sum_{0 \leq i \leq k} (-1)^i \frac{m(v\sigma)}{m(\sigma)} \phi(v\sigma_i) = \\
&= \sum_{\substack{\gamma \in \Sigma(k+1) \\ \sigma \subset \gamma}} \frac{m(\gamma)}{(k+2)!m(\sigma)} \phi(\sigma) - \sum_{\substack{v \in \Sigma(0) \\ v\sigma \in \Sigma(k+1)}} \sum_{0 \leq i \leq k} (-1)^i \frac{m(v\sigma)}{m(\sigma)} \phi(v\sigma_i) = \\
&= C_k \phi(\sigma) - \sum_{\substack{v \in \Sigma(0) \\ v\sigma \in \Sigma(k+1)}} \sum_{0 \leq i \leq k} (-1)^i \frac{m(v\sigma)}{m(\sigma)} \phi(v\sigma_i)
\end{aligned}$$

□

## 2.2 Localization

Let  $(v_0, \dots, v_j) = \tau \in \Sigma(j)$ , denote by  $X_\tau$  the *link* of  $\tau$  in  $X$ , that is, the subcomplex of dimension  $n - j - 1$  consisting on simplices  $\sigma = (w_0, \dots, w_k)$  such that  $\{v_0, \dots, v_j\}, \{w_0, \dots, w_k\}$  are disjoint as sets and  $(v_0, \dots, v_j, w_0, \dots, w_k) = \tau\sigma \in \Sigma(j+k+1)$ . The isotropy group  $\Gamma_\tau$  acts by automorphisms on  $X_\tau$  and if we denote by  $\rho_\tau$  the restriction of  $\rho$  to  $\Gamma_\tau$ , we get that  $\rho_\tau$  is a unitary representation of  $\Gamma_\tau$ . In this section we will always assume that  $j \geq r$ , which means that  $X_\tau$  is a finite simplicial complex.

1. For  $0 \leq k \leq n - j - 1$ , denote by  $\Sigma_\tau(k)$  the set of ordered  $k$ -simplices and choose a set  $\Sigma_\tau(k, \Gamma_\tau) \subseteq \Sigma_\tau(k)$  of representatives of  $\Gamma_\tau$ -orbits.
2. For a simplex  $\sigma \in \Sigma_\tau(k)$  denote by  $m_\tau(\sigma) = m(\tau\sigma)$  and by our definition,  $m_\tau(\sigma) > 0$  for every  $\sigma$ . Denote by  $C_{\tau,k}$  the constant such that for every  $\sigma \in \Sigma_\tau(k)$  one has

$$\sum_{\gamma \in \Sigma_\tau(k+1), \sigma \subset \gamma} m_\tau(\gamma) = (k+2)! C_{\tau,k} m_\tau(\sigma)$$

3. For a simplex  $\sigma \in \Sigma_\tau(k)$ , denote by  $\Gamma_{\tau\sigma}$  the stabilizer of  $\sigma$  in  $\Gamma_\tau$  (or the stabilizer of  $\tau\sigma$  in  $\Gamma$ ).
4. For  $0 \leq k \leq n - j - 1$ , denote by  $C^k(X_\tau, \rho_\tau)$  the space of simplicial  $k$ -cochains of  $X_\tau$  which are twisted by  $\rho_\tau$ .
5. On  $C^k(X_\tau, \rho_\tau)$  we define an Hermitian form as before, i.e. for every  $\phi, \psi \in C^k(X_\tau, \rho_\tau)$ :

$$\langle \phi, \psi \rangle := \sum_{\sigma \in \Sigma_\tau(k, \Gamma_\tau)} \frac{m_\tau(\sigma)}{(k+1)! |\Gamma_{\tau\sigma}|} \langle \phi(\sigma), \psi(\sigma) \rangle$$

note that for every  $\phi \in C^k(X_\tau, \rho_\tau)$  we have:

$$\|\phi\|^2 = \sum_{\sigma \in \Sigma_\tau(k, \Gamma_\tau)} \frac{m_\tau(\sigma)}{(k+1)! |\Gamma_\tau \sigma|} |\phi(\sigma)|^2 = \frac{1}{|\Gamma_\tau|} \sum_{\sigma \in \Sigma_\tau(k)} \frac{m(\tau\sigma)}{(k+1)!} |\phi(\sigma)|^2$$

6. On  $C^k(X_\tau, \rho_\tau)$  the differential is defined as before and denoted by  $d_\tau$  (where  $\delta_\tau = d_\tau^*$ ,  $\Delta_\tau^+ = \delta_\tau d_\tau$ ,  $\Delta_\tau^- = d_\tau \delta_\tau$ ).

**Proposition 2.6.** *For  $\tau$  of dimension  $j \geq r$  for every  $0 \leq k \leq n - j - 2$ , we have  $C_{\tau, k} = C_{j+k+1}$*

*Proof.* For  $\sigma \in \Sigma_\tau(k)$  we have by definition

$$\begin{aligned} (k+2)! C_{\tau, k} m_\tau(\sigma) &= \sum_{\substack{\gamma \in \Sigma_\tau(k+1) \\ \sigma \subset \gamma}} m_\tau(\gamma) = \\ &= \sum_{\substack{\gamma \in \Sigma_\tau(k+1) \\ \sigma \subset \gamma}} m(\tau\gamma) = \sum_{\substack{\eta \in \Sigma(j+k+2) \\ \tau\sigma \subset \eta}} \frac{(k+2)!}{(j+k+3)!} m(\eta) = \\ &= (j+1)!(k+2)! C_{j+k+1} m(\tau\sigma) = (k+2)! C_{j+k+1} m_\tau(\sigma) \end{aligned}$$

□

We also define  $C^k(X_\tau, \rho_\tau)$ ,  $d_\tau$ ,  $\delta_\tau$ ,  $\Delta_\tau^+$ ,  $\Delta_\tau^-$ ,  $\Delta_\tau$   
Define the *localization map*

$$C^k(X, \rho) \rightarrow C^{k-j-1}(X_\tau, \rho_\tau), \quad \phi \rightarrow \phi_\tau$$

Where  $\phi_\tau(\sigma) = \phi(\tau\sigma)$ .

**Proposition 2.7.** *For every  $\phi \in C^k(X, \rho)$ ,  $k > r$  one has:*

1.

$$\sum_{\tau \in \Sigma(k-1, \Gamma)} \|\phi_\tau\|^2 = (k+1)! \|\phi\|^2$$

2.

$$\sum_{\tau \in \Sigma(k-1, \Gamma)} \|\phi_\tau^0\|^2 = \frac{k!}{C_{k-1}} \|\delta\phi\|^2$$

where  $\phi_\tau^0$  is the projection of  $\phi_\tau$  on the space of constant functions.

*Proof.* 1. This proposition is proved in [BS97, Lemma 1.10] and since the proof doesn't take the weights into account we will not repeat it here.



2. For every  $\phi \in C^k(X, \rho)$  and every  $\tau \in \Sigma(k-1, \Gamma)$  one has:

$$\phi_\tau^0 = \frac{1}{\sum_{v \in \Sigma_\tau(0)} m(\tau v)} \sum_{v \in \Sigma_\tau(0)} m(\tau v) \phi(\tau v)$$

Note that

$$\sum_{v \in \Sigma_\tau(0)} m(\tau v) = \frac{1}{(k+1)!} \sum_{\sigma \in \Sigma(k), \tau \subset \sigma} m(\sigma) = C_{k-1} m(\tau)$$

therefore we get

$$\phi_\tau^0 = \frac{1}{C_{k-1} m(\tau)} \sum_{v \in \Sigma_\tau(0)} m(\tau v) \phi(\tau v) = \frac{(-1)^k}{C_{k-1}} \delta(\tau)$$

and therefore

$$\begin{aligned} \|\phi_\tau^0\|^2 &= \frac{1}{|\Gamma_\tau|} \sum_{v \in \Sigma_\tau(0)} m(\tau v) \left| \frac{(-1)^k}{C_{k-1}} \delta(\tau) \right|^2 = \\ &= \frac{1}{|\Gamma_\tau|} \frac{m(\tau)}{C_{k-1}} |\delta(\tau)|^2 \end{aligned}$$

and the equality in the proposition follows.  $\square$

In cases where the links finite and the stabilizers of the links are compact we have the following proposition which is proven in [BS97] (the proposition is general and does not depend on the weights so we will not prove it here):

**Proposition 2.8.** ([BS97, Lemma 2.3]) *For a simplex  $\tau$  of dimension  $j \geq r$ , let  $\alpha^+$  is the smallest positive eigenvalue of  $\Delta_\tau^+$  on  $C^0(X_\tau, \mathbb{R})$ , which is the space of (untwisted) 0-cochains on  $X_\tau$  with values in  $\mathbb{R}$  (this can be thought of the space of representations in  $\mathbb{R}$ , where the group is not  $\Gamma_\tau$  but the trivial group). Then  $\|\Delta_\tau^+ \phi\| \geq \alpha^+ \|\phi\|$  for all  $\phi \in C^0(X_\tau, \rho_\tau)$  perpendicular to  $\ker(\Delta_\tau^+)$  (Note that  $X_\tau$  and  $\Gamma_\tau$  are compact).*

*From now on denote by  $\lambda(X_\tau) = \alpha^+$  the smallest positive eigenvalue of  $\Delta_\tau^+$  on  $C^0(X_\tau, \mathbb{R})$ .*

## 2.3 Restriction

Let  $Y$  is a finite simplicial complex of dimension  $s$  with connected links with a weight function  $m$ ,  $\Lambda$  is a compact group acting on  $Y$  and  $\varrho$  is a unitary representation of  $\Lambda$  (we use different notation to stress that the assumptions on  $Y$  are stronger than the assumptions on  $X$ ).

**Definition 2.9.** *For  $\phi \in C^k(Y, \varrho)$  and  $\tau \in \Sigma(l)$  s.t.  $k+l+1 \leq s$ , the restriction of  $\phi$  to  $Y_\tau$  is a function  $\phi^\tau \in C^k(Y_\tau, \varrho_\tau)$  defined as follows:*

$$\forall \sigma \in \Sigma_\tau(k), \phi^\tau(\sigma) = \phi(\sigma)$$

**Lemma 2.10.** *Let  $\phi \in C^k(Y, \varrho)$  then*

$$C_k \|\phi\|^2 = \sum_{u \in \Sigma(0, \Lambda)} \|\phi^u\|^2$$

*Proof.*

$$\begin{aligned} \sum_{u \in \Sigma(0, \Lambda)} \|\phi^u\|^2 &= \sum_{u \in \Sigma(0, \Lambda)} \sum_{\tau \in \Sigma_u(k, \Lambda_u)} \frac{m_u(\tau)}{(k+1)!|\Lambda_{u\tau}|} |\phi^u(\tau)|^2 = \\ &= \sum_{u \in \Sigma(0, \Lambda)} \frac{1}{(k+1)!|\Lambda_u|} \sum_{\tau \in \Sigma_u(k)} m(u\tau) |\phi(\tau)|^2 = \\ &= \sum_{u \in \Sigma(0, \Lambda)} \frac{1}{(k+1)!|\Lambda_u|} \sum_{\gamma \in \Sigma(k+1), u \subset \gamma} \frac{1}{k+2} m(\gamma) |\phi(\gamma - u)|^2 \end{aligned}$$

where  $\gamma - u$  means deleting the vertex of  $u$  from  $\gamma$ . Changing the order of summation gives

$$\begin{aligned} &\sum_{\gamma \in \Sigma(k+1, \Lambda)} \frac{m(\gamma)}{(k+2)!|\Lambda_\gamma|} \sum_{u \in \Sigma(0), u \subset \gamma} |\phi(\gamma - u)|^2 = \\ &= \sum_{\gamma \in \Sigma(k+1, \Lambda)} \frac{m(\gamma)}{(k+2)!|\Lambda_\gamma|} \sum_{\sigma \in \Sigma(k), \sigma \subset \gamma} \frac{1}{(k+1)!} |\phi(\sigma)|^2 = \\ &= \sum_{\sigma \in \Sigma(k, \Lambda)} \frac{|\phi(\sigma)|^2}{(k+2)!(k+1)!|\Lambda_\sigma|} \sum_{\gamma \in \Sigma(k+1), \sigma \subset \gamma} m(\gamma) \end{aligned}$$

so we get

$$C_k \sum_{\sigma \in \Sigma(k, \Gamma)} \frac{m(\sigma) |\phi(\sigma)|^2}{(k+1)!|\Lambda_\sigma|} = C_k \|\phi\|^2$$

□

**Proposition 2.11.** *Let  $\phi \in C^0(Y, \varrho)$  then*

$$C_1 \|d\phi\|^2 = \sum_{u \in \Sigma(0, \Lambda)} \|d_u \phi^u\|^2$$

where  $d_u$  is the restriction of  $d$  to the link of  $u$ .

*Proof.* Note that

$$\forall (v_0, v_1) \in \Sigma_u(1), d_u \phi^u((v_0, v_1)) = \phi(v_0) - \phi(v_1) = d\phi((v_0, v_1)) = (d\phi)^u((v_0, v_1))$$

therefore  $d_u(\phi^u) = (d\phi)^u$  and the proposition follows. □

### 3 Criteria for vanishing cohomology

In [BS97, section 2] it is shown that if there is an  $\varepsilon > 0$  such that for every  $\phi \in C^k(X, \rho)$ ,  $d\phi = 0$  one has  $\|\delta\phi\|^2 \geq \varepsilon\|\phi\|^2$  then  $H^k(X, \rho) = 0$  and when  $X$  is contractible and the action of  $\Gamma$  is proper and cocompact we get that  $H^k(\Gamma, \rho) = 0$ . This section has three parts - first we will give a geometrical criterion for the above condition to hold which generalizes the criteria given [BS97] to the framework of general weights. Second, we will show that the fulfilment of the criterion for some  $k \geq r + 1$  implies that it is fulfilled for every  $j$  s.t.  $k \geq j \geq r + 1$ . Third we will show that the fulfilment of the criterion for some  $k \geq r + 1$  implies property (T).

We remind the reader that throughout this paper,  $X$  is a simplicial complex of dimension  $n$  such that all the links of  $X$  are connected and there is a constant  $0 \leq r \leq n - 2$  such that all the links of simplices of dimension  $r$  are finite. Also  $\Gamma$  is a locally compact, properly discontinuous, unimodular group of automorphisms of  $X$  acting cocompactly on  $X$  and  $\rho$  is a unitary representation of  $\Gamma$  on a complex Hilbert space  $H$ .

#### 3.1 A criterion for cohomology vanishing

**Lemma 3.1.** *For every  $r + 1 \leq k \leq n - 1$  and every  $\phi \in C^k(X, \rho)$  we have that*

$$k!\|d\phi\|^2 = \sum_{\tau \in \Sigma(k-1, \Gamma)} \left( \|d_\tau \phi_\tau\|^2 - C_k \frac{k}{k+1} \|\phi_\tau\|^2 \right)$$

*Proof.* For  $(v_0, \dots, v_{k+1}) = \sigma \in \Sigma(k+1)$  and  $0 \leq i < j \leq k+1$  denote

$$\sigma_{ij} = (v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_{k+1})$$

Then for every  $\phi \in C^k(X, \rho)$  we have

$$\begin{aligned} |d\phi(\sigma)|^2 &= \sum_{0 \leq i < j \leq k+1} |\phi_{\sigma_{ij}}(v_i) - \phi_{\sigma_{ij}}(v_j)|^2 - k \sum_{0 \leq i \leq k+1} |\phi(\sigma_i)|^2 = \\ &= \sum_{0 \leq i < j \leq k+1} \left( |\phi_{\sigma_{ij}}(v_i) - \phi_{\sigma_{ij}}(v_j)|^2 - \left(\frac{k}{k+1}\right) |\phi_{\sigma_{ij}}(v_j)|^2 \right) = \\ &= \frac{1}{k!} \sum_{\tau \in \Sigma(k-1), \tau \subset \sigma} \left( |d\phi_\tau(\sigma - \tau)|^2 - \frac{k}{k+1} \sum_{v \in \Sigma_\tau(0), v \subset \sigma - \tau} |\phi_\tau(v)|^2 \right) \end{aligned}$$

where  $\sigma - \tau$  is the 1 dimensional simplex obtained by deleting the the vertices of  $\tau$  from  $\sigma$ .

Now we can use this equality to connect  $\|d\phi\|$  to  $\|d_\tau \phi_\tau\|$  and  $\|\phi_\tau\|$ :

$$k!\|d\phi\|^2 = \sum_{\sigma \in \Sigma(k+1, \Gamma)} \frac{m(\sigma)}{(k+2)!|\Gamma_\sigma|} |d\phi(\sigma)|^2 =$$

$$\begin{aligned}
&= \sum_{\sigma \in \Sigma(k+1, \Gamma)} \frac{1}{(k+2)!|\Gamma_\sigma|} \sum_{\tau \in \Sigma(k-1), \tau \subset \sigma} m_\tau(\sigma - \tau) \left( |d\phi_\tau(\sigma - \tau)|^2 - \frac{k}{k+1} \sum_{v \in \Sigma_\tau(0), v \subset \sigma - \tau} |\phi_\tau(v)|^2 \right) = \\
&= \sum_{\tau \in \Sigma(k-1, \Gamma)} \frac{1}{(k+2)!|\Gamma_\tau|} \sum_{\sigma \in \Sigma(k+1), \tau \subset \sigma} m_\tau(\sigma - \tau) \left( |d\phi_\tau(\sigma - \tau)|^2 - \frac{k}{k+1} \sum_{v \in \Sigma_\tau(0), v \subset \sigma - \tau} |\phi_\tau(v)|^2 \right) = \\
&= \sum_{\tau \in \Sigma(k-1, \Gamma)} \frac{1}{(k+2)!|\Gamma_\tau|} \frac{(k+2)!}{2!} \sum_{\eta \in \Sigma_\tau(1)} m_\tau(\eta) \left( |d\phi_\tau(\eta)|^2 - \frac{k}{k+1} \sum_{v \in \Sigma_\tau(0), v \subset \eta} |\phi_\tau(v)|^2 \right) = \\
&= \sum_{\tau \in \Sigma(k-1, \Gamma)} \frac{1}{|\Gamma_\tau|} \sum_{\eta \in \Sigma_\tau(1)} \frac{m_\tau(\eta)}{2} |d\phi_\tau(\eta)|^2 - \\
&\quad - \frac{k}{k+1} \sum_{\tau \in \Sigma(k-1, \Gamma)} \frac{1}{|\Gamma_\tau|} \sum_{\eta \in \Sigma_\tau(1)} \sum_{v \in \Sigma_\tau(0), v \subset \eta} \frac{m_\tau(\eta)}{2} |\phi_\tau(v)|^2
\end{aligned}$$

Note that

$$\sum_{\tau \in \Sigma(k-1, \Gamma)} \frac{1}{|\Gamma_\tau|} \sum_{\eta \in \Sigma_\tau(1)} \frac{m_\tau(\eta)}{2} |d\phi_\tau(\eta)|^2 = \sum_{\tau \in \Sigma(k-1, \Gamma)} \|d_\tau \phi_\tau\|^2$$

and that

$$\begin{aligned}
&\frac{k}{k+1} \sum_{\tau \in \Sigma(k-1, \Gamma)} \frac{1}{|\Gamma_\tau|} \sum_{\eta \in \Sigma_\tau(1)} \sum_{v \in \Sigma_\tau(0), v \subset \eta} \frac{m_\tau(\eta)}{2} |\phi_\tau(v)|^2 = \\
&= \frac{k}{k+1} \sum_{\tau \in \Sigma(k-1, \Gamma)} \frac{1}{|\Gamma_\tau|} \sum_{v \in \Sigma_\tau(0)} |\phi_\tau(v)|^2 \sum_{\eta \in \Sigma_\tau(1), v \subset \eta} \frac{m_\tau(\eta)}{2} = \\
&= \frac{k}{k+1} \sum_{\tau \in \Sigma(k-1, \Gamma)} \frac{1}{|\Gamma_\tau|} \sum_{v \in \Sigma_\tau(0)} |\phi_\tau(v)|^2 m_\tau(v) C_{\tau,0} = \sum_{\tau \in \Sigma(k-1, \Gamma)} C_k \frac{k}{k+1} \|\phi_\tau\|^2
\end{aligned}$$

So we get the desired equality.  $\square$

Now we prove a version of [BS97, Theorem 2.5] in our setting:

**Theorem 3.2.** *Let  $r+1 \leq k \leq n-1$ . Assume that there is a constant  $\lambda > C_k \frac{k}{k+1}$  such that for every  $\tau \in \Sigma(k-1)$  we have that  $\lambda(X_\tau) \geq \lambda$  then there is an  $\varepsilon > 0$  such that for every  $\phi \in C^k(X, \rho)$ ,  $d\phi = 0$  one has*

$$\|\delta\phi\|^2 \geq \varepsilon \|\phi\|^2$$

and therefore  $H^k(X, \rho) = H^k(\Gamma, \rho) = 0$ .

*Proof.* By the above lemma, for every  $\phi \in C^k(X, \rho)$  we have

$$k! \|d\phi\|^2 = \sum_{\tau \in \Sigma(k-1, \Gamma)} \left( \|d_\tau \phi_\tau\|^2 - C_k \frac{k}{k+1} \|\phi_\tau\|^2 \right)$$

For every  $\tau \in \Sigma(k-1)$  link  $X_\tau$  is connected and finite so the kernel of  $d_\tau$  consists of constant maps and we get that

$$\|d_\tau \phi_\tau\|^2 \geq \lambda \|\phi_\tau\|^2 - \lambda \|\phi_\tau^0\|^2$$

Therefore

$$k! \|d\phi\|^2 \geq \sum_{\tau \in \Sigma(k-1, \Gamma)} \left( (\lambda - C_k \frac{k}{k+1}) \|\phi_\tau\|^2 - \lambda \|\phi_\tau^0\|^2 \right)$$

Recall that

$$\sum_{\tau \in \Sigma(k-1, \Gamma)} \|\phi_\tau\|^2 = (k+1)! \|\phi\|^2$$

and that

$$\sum_{\tau \in \Sigma(k-1, \Gamma)} \|\phi_\tau^0\|^2 = \frac{k!}{C_{k-1}} \|\delta\phi\|^2$$

So we get that

$$k! \|d\phi\|^2 \geq (k+1)! (\lambda - C_k \frac{k}{k+1}) \|\phi\|^2 - \lambda \frac{k!}{C_{k-1}} \|\delta\phi\|^2$$

For  $\phi \in \ker(d)$  this reads

$$\|\delta\phi\|^2 \geq \frac{C_{k-1}(k+1)}{\lambda} (\lambda - C_k \frac{k}{k+1}) \|\phi\|^2$$

So for  $\varepsilon = \frac{C_{k-1}(k+1)}{\lambda} (\lambda - C_k \frac{k}{k+1}) > 0$  we get the desired inequality.  $\square$

### 3.2 A stronger vanishing result

In this section we will show the following theorem:

**Theorem 3.3.** *Let  $r+1 < k \leq n-1$ . Assume that there is a constant  $\lambda_k > C_k \frac{k}{k+1}$  such that for every  $\sigma \in \Sigma(k-1)$  we have that  $\lambda(X_\sigma) \geq \lambda_k$ , then for every  $r+1 \leq j < k \leq n-1$  there is a constant  $\lambda_j > C_j \frac{j}{j+1}$  such that for every  $\tau \in \Sigma(j-1)$  we have that  $\lambda(X_\tau) \geq \lambda_j$ .*

Combined with the theorem 3.2 in the previous section this implies the following corollary which is the first implication of Theorem 1.1 in the introduction when  $m$  is the weight function given in [BS97]:

**Corollary 3.4.** *Let  $r + 1 \leq k \leq n - 1$ . Assume that there is a constant  $\lambda_k > C_k \frac{k}{k+1}$  such that for every  $\tau \in \Sigma(k-1)$  we have that  $\lambda(X_\tau) \geq \lambda_k$ , then for every  $r + 1 \leq j \leq k \leq n - 1$  we have that  $H^j(X, \rho) = H^j(\Gamma, \rho) = 0$ .*

Fix  $r < j \leq n - 2$  and  $\tau \in \Sigma(j-1)$ . Note that  $X_\tau$  is finite and that  $\Gamma$  is compact and so we apply the results about restriction to  $X_\tau$ . Denote by  $\lambda_\eta = \lambda(X_\eta, \rho_\eta)$  the smallest positive eigenvalue of  $\Delta_\eta^+$ .

**Proposition 3.5.** *If for every  $v \in \Sigma_\tau(0)$ , we have  $\lambda_{\tau v} \geq \mu$ , then*

$$\lambda_\tau \geq \frac{C_{j+1}(-C_{j+2} + 2\mu)}{\mu}$$

*Proof.* Recall that for  $\phi \in C^0(X_\tau, \rho_\tau)$

$$\Delta_\tau^+ \phi(u) = C_{\tau,0} \phi(u) - \frac{1}{m_\tau(u)} \sum_{(u,v) \in \Sigma_\tau(1)} m_\tau((u,v)) \phi(v)$$

Take  $\phi \in C^0(X_\tau, \rho_\tau)$  to be an eigenfunction of  $\lambda_\tau$ , i.e.

$$\Delta_\tau^+ \phi(u) = \lambda_\tau \phi(u)$$

Fix  $v \in \Sigma_\tau(0)$  and denote by  $(\phi^v)^0$  the projection of  $\phi^v$  to the space of constant maps on  $X_{\tau v}$  and by  $(\phi^v)^1$  its orthogonal compliment. Explicitly we have

$$\begin{aligned} \frac{1}{|\Gamma_{\tau v}|} \sum_{u \in \Sigma_{\tau v}(0)} m_{\tau v}(u) \phi^v(u) &= \frac{1}{|\Gamma_{\tau v}|} \sum_{u \in \Sigma_{\tau v}(0)} m_{\tau v}(u) (\phi^v)^0 = \\ &= \frac{(\phi^v)^0}{|\Gamma_{\tau v}|} \sum_{u \in \Sigma_{\tau v}(0)} m_{\tau v}(u) = \frac{(\phi^v)^0}{|\Gamma_{\tau v}|} \sum_{u \in \Sigma_\tau(0), (v,u) \in \Sigma_\tau(1)} m_\tau((v,u)) = \frac{(\phi^v)^0}{|\Gamma_v|} C_{\tau,0} m_\tau(v) \end{aligned}$$

Therefore

$$(\phi^v)^0 = \frac{1}{C_{\tau,0} m_\tau(v)} \sum_{u \in \Sigma_{\tau v}(0)} m_{\tau v}(u) \phi^v(u)$$

and

$$(\phi^v)^1 = \phi^v - (\phi^v)^0$$

Note that since  $\Delta_\tau^+ \phi(v) = \lambda_\tau \phi(v)$ , we get that

$$\begin{aligned} \lambda_\tau \phi(v) &= C_{\tau,0} \phi(v) - \frac{1}{m_\tau(v)} \sum_{(v,u) \in \Sigma_\tau(1)} m_\tau((v,u)) \phi(u) = \\ &= C_{\tau,0} \phi(v) - \frac{1}{m_\tau(v)} \sum_{u \in \Sigma_{\tau v}(0)} m_{\tau v}(u) \phi^v(u) \end{aligned}$$

Therefore

$$(\phi^v)^0 = \frac{C_{\tau,0} - \lambda_\tau}{C_{\tau,0}} \phi(v)$$

Since  $X_{\tau v}$  is connected for every  $v \in \Sigma_\tau(0)$ , the kernel of  $\Delta_{\tau v}^+$  is the space of constant maps. Therefore by the definition of  $\lambda_{\tau v}$ , we have for every  $v \in \Sigma_\tau(0)$

$$\|d_{\tau v}\phi^v\|^2 \geq \lambda_{\tau v}\|(\phi^v)^1\|^2 = \lambda_{\tau v}\|(\phi^v)\|^2 - \lambda_{\tau v}\|(\phi^v)^0\|^2$$

Recall that

$$C_{\tau,1}\|d_\tau\phi\|^2 = \sum_{v \in \Sigma_\tau(0, \Gamma_\tau)} \|d_{\tau v}\phi^v\|^2$$

So we get

$$C_{\tau,1}\|d_\tau\phi\|^2 \geq \sum_{v \in \Sigma_\tau(0, \Gamma_\tau)} \lambda_{\tau v}\|(\phi^v)\|^2 - \lambda_{\tau v}\|(\phi^v)^0\|^2$$

Note that

$$\begin{aligned} \|(\phi^v)^0\|^2 &= \frac{1}{|\Gamma_{\tau v}|} \sum_{u \in \Sigma_{\tau v}(0)} m_{\tau v}(u)|(\phi^v)^0|^2 = \\ &= \frac{C_{\tau,0}m_\tau(v)}{|\Gamma_{\tau v}|}|(\phi^v)^0|^2 = \frac{m_\tau(v)}{|\Gamma_{\tau v}|} \frac{(C_{\tau,0} - \lambda_\tau)^2}{C_{\tau,0}} |\phi(v)|^2 \end{aligned}$$

So we get

$$\begin{aligned} C_{\tau,1}\lambda_\tau\|\phi\|^2 &= C_{\tau,1}\|d\phi\|^2 \geq \\ &\geq \sum_{v \in \Sigma_\tau(0, \Gamma_\tau)} \lambda_{\tau v} \left( \|(\phi^v)\|^2 - \frac{m_\tau(v)}{|\Gamma_{\tau v}|} \frac{(C_{\tau,0} - \lambda_\tau)^2}{C_{\tau,0}} |\phi(v)|^2 \right) \geq \\ &\geq \sum_{v \in \Sigma_\tau(0, \Gamma_\tau)} \mu \left( \|(\phi^v)\|^2 - \frac{m_\tau(v)}{|\Gamma_{\tau v}|} \frac{(C_{\tau,0} - \lambda_\tau)^2}{C_{\tau,0}} |\phi(v)|^2 \right) = \\ &= \mu(C_{\tau,0} - \frac{(C_{\tau,0} - \lambda_\tau)^2}{C_{\tau,0}}) \|\phi\|^2 \end{aligned}$$

Therefore

$$C_{\tau,1}\lambda_\tau \geq \mu(C_{\tau,0} - \frac{(C_{\tau,0} - \lambda_\tau)^2}{C_{\tau,0}})$$

Which yields

$$\lambda_\tau(\lambda_\tau \frac{\mu}{C_{\tau,0}} - 2\mu + C_{\tau,1}) \geq 0$$

and since  $\lambda_\tau > 0$  we get

$$\lambda_\tau \geq \frac{C_{\tau,0}(-C_{\tau,1} + 2\mu)}{\mu}$$

Recall that  $C_{\tau,0} = C_j$ ,  $C_{\tau,1} = C_{j+1}$  and therefore

$$\lambda_\tau \geq \frac{C_j(-C_{j+1} + 2\mu)}{\mu}$$

□

Now we can generalize the above proposition to:

**Proposition 3.6.** *For  $0 \leq l \leq n - j - 3$ , if for all  $\eta \in \Sigma_\tau(l)$  we have  $\lambda_{\tau\eta} \geq \mu$  then*

$$\lambda_\tau \geq \frac{C_j(-(l+1)C_{j+1+l} + (l+2)\mu)}{-lC_{j+1+l} + (l+1)\mu}$$

*Proof.* We will prove this proposition by induction. For  $l = 0$  the proposition was proven above. Now we assume it is true for  $l$  and prove it for  $l + 1$ : assume that for every  $\eta \in \Sigma_\tau(l+1)$  we have that  $\lambda_{\tau\eta} \geq \mu$ , then by the above proposition 3.5 we get that for every  $\gamma \in \Sigma_\tau(k)$  we have

$$\lambda_{\tau\gamma} \geq \frac{C_{j+1+l}(-C_{j+1+l} + 2\mu)}{\mu}$$

Denote  $\mu' = \frac{C_{j+1+l}(-C_{j+2+l} + 2\mu)}{\mu}$ . From the induction assumption

$$\begin{aligned} \lambda_\tau &\geq \frac{C_j(-(l+1)C_{j+1+l} + (l+2)\mu')}{-lC_{j+1+l} + (l+1)\mu'} = \\ &= \frac{C_j(-(l+2)C_{j+2+l} + (l+3)\mu)}{-(l+1)C_{j+2+l} + (l+2)\mu} \end{aligned}$$

□

Now we can prove 3.3:

*Proof.* Let  $r + 1 < k \leq n - 1$ . Assume that there is a constant  $\lambda_k > C_k \frac{k}{k+1}$  such that for every  $\sigma \in \Sigma(k-1)$  we have that  $\lambda(X_\sigma) \geq \lambda_k$ . We want to prove that for every  $r + 1 \leq j < k \leq n - 1$  there is a constant  $\lambda_j > C_j \frac{j}{j+1}$  such that for every  $\tau \in \Sigma(j-1)$  we have that  $\lambda(X_\tau) \geq \lambda_j$ . If  $k = j$  the theorem is obvious, so assume  $k < j$ . Fix  $\tau \in \Sigma(j-1)$  so for every  $\eta \in \Sigma_\tau(k-j-1)$  we have that  $\tau\eta$  is a simple of dimension  $k-1$  and therefore  $\lambda_{\tau\eta} \geq \lambda_k > C_k \frac{k}{k+1}$ . Define the function:

$$f(\mu) = \frac{C_j(-(k-j)C_k + (k-j+1)\mu)}{-(k-j-1)C_k + (k-j)\mu}$$

by differentiation it is easy to see that this function is strictly monotone increasing so if  $\lambda_k > C_k \frac{k}{k+1}$  then  $f(\lambda_k) > f(C_k \frac{k}{k+1})$ . By the above proposition 3.6 we have that

$$\lambda(X_\tau) \geq f(\lambda_k)$$

and therefore

$$\lambda(X_\tau) > f(C_k \frac{k}{k+1}) = C_j \frac{j}{j+1}$$



Now take

$$\lambda_j = \min_{\tau \in \Sigma(j-1, \Gamma)} \lambda(X_\tau) > C_j \frac{j}{j+1}$$

□

### 3.3 Property (T)

In the subsection we will prove that every one of the above criteria given for the vanishing of some cohomology implies property (T). Namely:

**Theorem 3.7.** *Let  $r+1 \leq k \leq n-1$ . Assume that there is a constant  $\lambda > C_k \frac{k}{k+1}$  such that for every  $\tau \in \Sigma(k-1)$  we have that  $\lambda(X_\tau) \geq \lambda$  then  $\Gamma$  has property (T).*

**Remark 3.8.** *Note that the theorem stated above implies the the second implication of Theorem 1.1 in the introduction when  $m$  is taken to be the weight function given in [BS97].*

**Remark 3.9.** *If the link of every vertex in  $X$  is compact this theorem follows directly from theorem 3.3.*

First let us state this proposition which says that when proving property (T) one does not have to act on a contractible simplicial complex:

**Proposition 3.10.** *Let  $X'$  is a simplicial complex of dimension  $n$  such that all the links of  $X'$  are connected and there is a constant and such that all the links of vertices are finite. Let  $\Gamma$  is a locally compact, properly discontinuous, unimodular group of automorphisms of  $X'$  acting cocompactly on  $X'$ . If for every unitary representation  $\rho'$  without an invariant vector there is a constant  $\varepsilon > 0$  such that for every  $\phi \in C^1(X', \rho), \phi \in \ker(d)$  the following inequality holds*

$$\|\delta\phi\|^2 \geq \varepsilon\|\phi\|^2$$

*Then  $\Gamma$  has property (T).*

*Proof.* To distinguish the differentials in this proof we shall denote

$$d_0 : C^0(X', \rho) \rightarrow C^1(X', \rho)$$

$$d_1 : C^1(X', \rho) \rightarrow C^2(X', \rho)$$

and  $\delta = d_1^*$ .

To show that  $\Gamma$  has property (T) it is enough to show that for every unitary representation  $\rho$  without an invariant vector there is  $\varepsilon > 0$  s.t. for every  $\psi \in C^0(X', \rho)$  the following inequality holds:

$$\|d_0\psi\|^2 \geq \varepsilon\|\psi\|^2$$

(for proof see [BdlHV08, Proposition 5.4.5])(note that since  $\rho$  doesn't have an invariant vector,  $\psi$  can not be a map constant).

Take  $d_0\psi = \phi \in \ker(d_1)$ , since  $\delta d_0 = \Delta^+$  we get

$$\|\Delta^+\psi\|^2 \geq \varepsilon \langle \Delta^+\psi, \psi \rangle$$

or

$$\langle d_0(\Delta^+)^{\frac{1}{2}}\psi, d_0(\Delta^+)^{\frac{1}{2}}\psi \rangle \geq \varepsilon \langle (\Delta^+)^{\frac{1}{2}}\psi, (\Delta^+)^{\frac{1}{2}}\psi \rangle$$

Since  $\ker((\Delta^+)^{\frac{1}{2}}) = \{0\}$  then  $\overline{\text{im}((\Delta^+)^{\frac{1}{2}})} = (\ker((\Delta^+)^{\frac{1}{2}}))^{\perp} = H$  and therefore for every  $\psi \in C^0(X', \rho)$  the inequality

$$\|d_0\psi\|^2 \geq \varepsilon \|\psi\|^2$$

holds. □

Second, to prove the above theorem, we will translate it to a matrix eigenvalue problem. To do this we'll need the following lemma (which is a standard result concerning normalized weighted Laplacian except for the part of  $C_0$ ):

**Lemma 3.11.** *Let  $Z = (V, E)$  be a connected finite graph with a weight function  $m$ . And  $\Delta^+ : C^0(Z, \mathbb{R}) \rightarrow C^0(Z, \mathbb{R})$  is defined as before (with respect to the weight function). Define the following  $|V| \times |V|$  matrix:*

$$(A_Z)(u, v) = \begin{cases} C_0 & u = v \\ \frac{-m((u, v))}{\sqrt{m(u)m(v)}} & (u, v) \in E \\ 0 & \text{otherwise} \end{cases}$$

*Then the eigenvalues of  $\Delta^+$  and of the matrix  $A_Z$  coincide and in particular  $\lambda(Z)$  is equal to the smallest positive eigenvalue of  $A_Z$ .*

*Proof.* Denote by  $(\cdot, \cdot)$  the standard inner product in  $\mathbb{R}^{|V|}$ , and let  $x = (x_v)$  be an eigenvector of  $A_Z$  with eigenvalue  $\mu$ . Define a new vector  $y = (y_v)$  as  $y_v = \sqrt{m(v)}x_v$ , then  $Ax = \mu x$  is equivalent to

$$\forall v \in V, C_0x_v - \sum_{u \in V, (v, u) \in E} \frac{m((u, v))}{\sqrt{m(u)m(v)}}x_u = \mu x_v$$

which is equivalent to

$$\forall v \in V, C_0 \frac{1}{\sqrt{m(v)}}y_v - \sum_{u \in V, (v, u) \in E} \frac{-m((u, v))}{m(u)\sqrt{m(v)}}y_u = \mu \frac{1}{\sqrt{m(v)}}y_v$$

Multiplying by  $\sqrt{m(v)}$  gives  $\Delta^+y = \mu y$  (and in the same way we can start with an eigenvector  $y$  of  $\Delta^+$  and get an eigenvector of  $A_Z$ ). □

Now we turn to prove the theorem stated in the beginning of this subsection:

*Proof.* To use 3.10 we will define a new simplicial complex  $X'$  on which  $\Gamma$  acts such that the conditions of 3.10 are fulfilled (in particular, the link of every vertex of  $X'$  should be finite):

Let  $r + 1 \leq k \leq n - 1$  such that there is a constant  $\lambda > C_k \frac{k}{k+1}$  such that for every  $\tau \in \Sigma(k-1)$  we have that  $\lambda(X_\tau) \geq \lambda$ . Define the following weighted 2 dimensional simplicial complex  $X'$ :

1. To avoid confusion, we will denote the ordered  $k$ -simplices of  $X'$  as  $\Sigma'(k)$  and the orbit representatives as  $\Sigma'(k, \Gamma)$ .
2. 0-simplices of  $X'$  are  $k-1$  (unordered) simplices of  $X$ . The weight of every such 0-simplex  $\sigma$  will be  $m'(\sigma) = m(\sigma)$  where  $m$  is the weight given to  $\sigma$  in  $X$ .
3. two different 0-simplices in  $X'$  are connected by an edge in  $X'$  if they are both contained in an  $k$  simplex in  $X$ , i.e. if  $\{u_0, \dots, u_{k-1}\}, \{v_0, \dots, v_{k-1}\}$  are  $k-1$ -simplices in  $X$ , then they are connected by an edge in  $X'$  if there is an  $k$  simplex in  $X$  which contain the vertices  $u_0, \dots, u_{k-1}, v_0, \dots, v_{k-1}$ . Note that a necessary condition for  $\{u_0, \dots, u_{k-1}\}, \{v_0, \dots, v_{k-1}\}$  to be connected by an edge is  $|\{u_0, \dots, u_{k-1}\} \cap \{v_0, \dots, v_{k-1}\}| = k-1$ . The weight of every such 1-simplex  $(\sigma_0, \sigma_1)$  will be  $m'(\sigma_0, \sigma_1) = m(\sigma_0 \cup \sigma_1)$  where  $m$  is the weight given to  $\sigma_0 \cup \sigma_1$  as a  $k$ -simplex in  $X$ .
4. A 2-simplex in  $X'$  is formed whenever 3 edges of  $X'$  bound a triangle. This happens in two cases:

(a) For 3 vertices of the form

$$\{u_0, \dots, u_{k-2}, v\}, \{u_0, \dots, u_{k-2}, w\}, \{u_0, \dots, u_{k-2}, x\}$$

where  $v, w, x$  are different from one another and  $\{u_0, \dots, u_{k-3}, v, w, x\}$  is an  $k+1$  dimensional simplex in  $X$ . We denote the set of simplices of this form as  $\Sigma'^{(1)}(2)$ . The weight of every such 2-simplex  $(\sigma_0, \sigma_1, \sigma_2) \in \Sigma'^{(1)}(2)$  will be  $m'(\sigma_0, \sigma_1, \sigma_2) = m(\sigma_0 \cup \sigma_1 \cup \sigma_2)$  where  $m$  is the weight given to  $\sigma_0 \cup \sigma_1 \cup \sigma_2$  as a  $k+1$  simplex in  $X$ .

(b) For 3 vertices of the form

$$\{u_0, \dots, u_{k-3}, v, w\}, \{u_0, \dots, u_{k-3}, w, x\}, \{u_0, \dots, u_{k-3}, v, x\}$$

where  $v, w, x$  are different from one another and  $\{u_0, \dots, u_{k-4}, v, w, x\}$  is an  $k$  dimensional simplex in  $X$ . We denote the set of simplices of this form as  $\Sigma'^{(2)}(2)$ . Note that for every  $(\sigma_0, \sigma_1) \in \Sigma'(1)$  there are exactly  $k-1$  different vertices  $\sigma_2 \in \Sigma'(0)$  such that  $(\sigma_0, \sigma_1, \sigma_2) \in \Sigma'(2)$ . The weight of every such 2-simplex  $(\sigma_0, \sigma_1, \sigma_2) \in \Sigma'^{(1)}(2)$  will be  $m'(\sigma_0, \sigma_1, \sigma_2) = \frac{\lambda m(\sigma_0 \cup \sigma_1 \cup \sigma_2)}{k}$  where  $m$  is the weight given to  $\sigma_0 \cup \sigma_1 \cup \sigma_2$  as a  $k$  simplex in  $X$ .

Since  $X$  is connected with connected links, we get that  $X'$  is connected with connected links and it is obvious that  $\Gamma$  acts on  $X'$  cocompactly and properly. Note that for every  $(\sigma_0, \sigma_1) \in \Sigma'(1)$  we have:

$$\begin{aligned} & \sum_{(\sigma_0, \sigma_1, \sigma_2) \in \Sigma'(2)} m'((\sigma_0, \sigma_1, \sigma_2)) = \\ &= \sum_{(\sigma_0, \sigma_1, \sigma_2) \in \Sigma'(1)(2)} m(\sigma_0 \cup \sigma_1 \cup \sigma_2) + \sum_{(\sigma_0, \sigma_1, \sigma_2) \in \Sigma'(2)(2)} \frac{\lambda}{k} m(\sigma_0 \cup \sigma_1 \cup \sigma_2) = \\ &= C_k m(\sigma_0 \cup \sigma_1) + \frac{\lambda(k-1)}{k} m(\sigma_0 \cup \sigma_1) = (C_k + \frac{\lambda(k-1)}{k}) m'((\sigma_0, \sigma_1)) \end{aligned}$$

Therefore  $C'_1 = C_k + \frac{\lambda(k-1)}{k}$ . Combining 3.2 and 3.10 we get that in order to show property (T), it will be enough to show that for every  $\sigma \in \Sigma'(0)$  we have

$$\lambda(X'_\sigma) > \frac{1}{2} (C_k + \frac{\lambda(k-1)}{k})$$

We will show that by showing that  $\lambda(X'_\sigma) \geq \lambda$ . Indeed if this holds, then it is enough to show that:

$$\lambda > \frac{1}{2} (C_k + \frac{\lambda(k-1)}{k})$$

which is equivalent to

$$\lambda > C_k \frac{(k-1)}{k}$$

which is assumed.

So we are left to show that for every  $\sigma \in \Sigma'(0)$  we have we have that  $\lambda(X'_\sigma) \geq \lambda$ . Denote by  $A_{X'_\sigma}$  the matrix corresponding to  $\Delta^+$  on  $X'_\sigma$  (as in lemma 3.11 above) and by  $A_{X_\sigma}$  the matrix corresponding to  $\Delta^+$  on  $X_\sigma$ . So we need to show that the smallest positive eigenvalue of  $A_{X'_\sigma}$  is larger or equal to  $\lambda$ . Let us write  $A_{X'_\sigma}$  explicitly:

$$A_{X'_\sigma}(\eta, \tau) = \begin{cases} C_k + \frac{\lambda(k-1)}{k} & \eta = \tau \\ -\frac{m(\sigma \cup \eta \cup \tau)}{\sqrt{m(\sigma \cup \eta)m(\sigma \cup \tau)}} & (\sigma, \eta, \tau) \in \Sigma'(1)(2) \\ -\frac{\lambda}{k} & (\sigma, \eta, \tau) \in \Sigma'(2)(2) \\ 0 & \text{otherwise} \end{cases}$$

(Note that in the cases  $(\sigma, \eta, \tau) \in \Sigma'(2)(2)$  we have that  $m(\sigma \cup \eta \cup \tau) = m(\sigma \cup \eta) = m(\sigma \cup \tau)$ ).

To complete the proof we need to recall some facts from matrix theory (see [Lan69] for further information and proofs):

1. The Kronecker product of matrices: for two matrices  $M = (a_{ij})$  which is a  $m \times m$  and  $Q$  which is a  $q \times q$  matrix the Kronecker product  $M \otimes Q$  is a  $mq \times mq$  matrix defined as

$$M \otimes Q = \begin{pmatrix} a_{11}Q & a_{12}Q & \dots & a_{1m}Q \\ a_{21}Q & a_{22}Q & \dots & a_{2m}Q \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}Q & a_{m2}Q & \dots & a_{mm}Q \end{pmatrix}$$

2. Denote by  $I_l$  the  $l \times l$  identity matrix. The Kronecker sum of  $M$  and  $Q$  as before is defined to be

$$M \oplus Q = I_q \otimes M + Q \otimes I_m$$

3. If  $M$  is a  $m \times m$  matrix with eigenvalues  $\lambda_1, \dots, \lambda_m$  and  $Q$  is a  $q \times q$  matrix with eigenvalues  $\mu_1, \dots, \mu_q$ , then  $M \oplus Q$  has the following  $mq$  eigenvalues:

$$\lambda_1 + \mu_1, \dots, \lambda_1 + \mu_q, \lambda_2 + \mu_1, \dots, \lambda_2 + \mu_q, \dots, \lambda_m + \mu_q$$

To finish the proof, notice that

$$A_{X'_\sigma} = A_{X_\sigma} \oplus B$$

Where  $B$  is the  $k \times k$  matrix defined as

$$B = \begin{pmatrix} \frac{\lambda(k-1)}{k} & -\frac{\lambda}{k} & \dots & -\frac{\lambda}{k} \\ -\frac{\lambda}{k} & \frac{\lambda(k-1)}{k} & \dots & -\frac{\lambda}{k} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{\lambda}{k} & -\frac{\lambda}{k} & \dots & \frac{\lambda(k-1)}{k} \end{pmatrix}$$

Note that  $B$  has the eigenvalue 0 with multiplicity 1 and all its other eigenvalues are  $\lambda$ , and  $A_{X_\sigma}$  has the eigenvalue 0 with multiplicity 1 and all its other eigenvalues are greater or equal to  $\lambda$ . Therefore  $A_{X_\sigma} \oplus B$  has eigenvalue 0 with multiplicity 1 and all its other eigenvalues are greater or equal to  $\lambda$ , therefore we are done.  $\square$

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